An Overview of Curvature

Ashwin Devaraj

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Contents

1 Introduction

1.1 Overview of Topics

These notes provide a gentle introduction to the fundamentals of differential geometry in Euclidean space, that is, the study of differentiable curves and regular surfaces. Differential geometry is largely concerned with the same problems as Euclidean geometry–namely how to measure lengths, angles, and areas–but done in a more general setting using the tools of calculus and linear algebra. These notes in particular most emphasize the notion of curvature and what information it provides about geometric objects. We will cover both the intuitive and rigorous definitions of curvature, and what properties of curves and surfaces can be determined from curvature. This discussion will culminate in Gauss' Theorema Egregium or "remarkable theorem."

These notes are primarily based on the content of Manfredo Do Carmo's *Differential Geometry of Curves and Surfaces*. We do not cover geometry on more abstract manifolds, which are addressed in his "sequel" textbook *Riemannian Geometry*.

1.2 Prerequisites

These notes assume knowledge of basic multivariable calculus (derivatives of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, the Jacobian matrix, etc.) as well as linear algebra. A formal real analysis background is not necessary, but one should be familiar with the definitions of open and closed sets in $\mathbb{R}^n$.

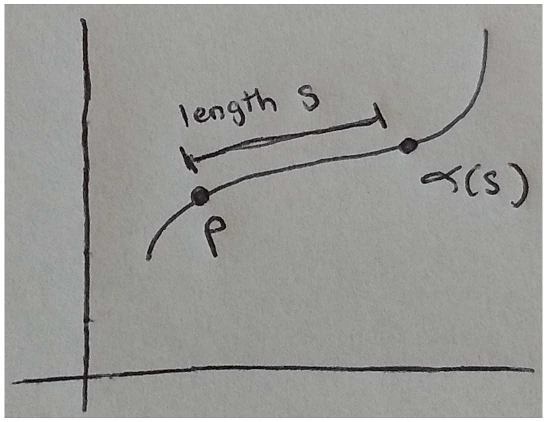


Figure 1: Parametrization of $\alpha : I \to \mathbb{R}^2$ by arc length

2 Curves

2.1 Basic Definitions

Definition 2.1. A curve in $\mathbb{R}^n$ is a smooth (infinitely differentiable) function from an open interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$.

Definition 2.2. A regular curve is a curve with a nonzero derivative over its domain.

For example, the function $f(t) = (\cos(t), \sin(t))$ is a regular curve in $\mathbb{R}^2$ whose image is the unit circle. Intuitively, a curve is just a smooth path in Euclidean space that we can trace without lifting our pen. A regular curve is one in which our pen never momentarily halts when tracing. t can thus be thought of as a "time" and $f(t)$ as the location an object is at time t.

Notice that different curves can have the same image. For example, $g(t) = (\cos(2t), \sin(2t))$ has the same unit circle image as $f(t)$ but is clearly a different function, and can be viewed as f except that when we trace it, we does so more quickly. f and g are considered to be 2 different *parametrizations* of the unit circle. A very important concept used to simplify many definitions and computations is the notion of parametrization by arc length. To understand this, we must first provide a definition of arc length.

Definition 2.3. The *arc length* of s a curve α from a point $t_0 \in I$ is defined as

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \, dt$$

where $\|\cdot\|$ denotes the standard Euclidean vector norm.

Intuitively, we can approximate the length of a curve by marking points on it and summing the lengths of the line segments connecting adjacent points. The above formula is the limiting sum as the distance between these points approaches 0.

Definition 2.4. A curve is $\alpha(t) : I \to \mathbb{R}^n$ *parametrized by arc length* if t is the arc length of α measured from some start point, so that $ds/dt = \|\alpha'(t)\| = 1$ and thus

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \, dt = t - t_0.$$

Simply put, a parametrization of a curve by arc length is just a function that takes an arc length with respect to a base point p on the curve and then outputs the corresponding point on the curve (see Figure 1). Such a parametrization is possible for any regular curve, so we are free to come up with definitions like curvature in the context of parametrizations by arc length.

Example 2.5. We now show an example in which a simple regular curve, that of a line, is rewritten as a parametrization by arc length. The regular curve $\alpha : I \to \mathbb{R}^3, \alpha(t) = x_0 + xt$, where I is an open interval, is a line that is not necessarily parametrized by arc length, since $\|\alpha'(t)\| = \|x\|$, which isn't necessarily equal to 1.

Our goal is to produce a new curve β that has the same image in $\mathbb{R}^3$ as α but instead takes as input an arc length measured from an initial point p_0 and then outputs the unique point p on the line such that the length of the line segment from p_0 to p is the input arc length. To accomplish this, we first pick an arbitrary $t_0 \in I$ and define $s(t)$ to be the arc length from $\alpha(t_0)$ to $\alpha(t)$ (with the "length" being negative when $t < t_0$):

$$s(t) = \int_{t_0}^{t} \|\alpha'(y)\| \, dy = \|x\|(t - t_0).$$

Since α is regular, $\alpha'(t) \neq 0$ and s is one-to-one (thus in bijection with its image $s(I)$). Informally, if a car starts driving at point $s(t_0)$ and never stops moving, then its distance traveled at every subsequent time step will be distinct. Thus the inverse of s, $t : s(I) \to I$ is well-defined and can easily be derived from the equation for $s(t)$:

$$t(s) = \frac{s}{\|x\|} + t_0.$$

Since $t(s)$ takes an arc length and outputs a value in I, we'd expect its composition with the original function α, $\beta(s) = \alpha \circ t(s) = x_0 + xt_0 + \frac{x}{\|x\|}s$ to be parametrized by arc length, and we show that this is in fact the case. $\beta(s(I)) = \alpha(t(s(I))) = \alpha(I)$, so the image of β is the same as that of α. Furthermore, $\|\beta'(s)\| = \|\alpha'(t) \cdot dt/ds\| = \|\beta'(s)\| = \|\alpha'(t) \cdot 1/\|\alpha'(t)\|\| = 1$, so β is indeed parametrized by arc length.

The above strategy of writing an arc length function s and composing its inverse with the original curve can be used to parametrize any regular curve by arc length.

2.2 Curvature

Definition 2.6. Let $\alpha(s)$ be a curve parametrized by arc length. Then the curvature of α at s is $k(s) = \|\alpha''(s)\|$.

Curvature can be viewed as the rate of change of the angle θ which the tangent line at a point s makes with some fixed reference line ℓ (see Figure 2), so the higher the curvature at a point, the sharper the curve "turns" at the point. This statement can be formalized as follows:

Theorem 2.7. *Let $\alpha(s)$ be a curve parametrized by arc length, and let v denote some reference unit vector in $\mathbb{R}^n$. Let $\theta(s)$ denote the angle the tangent vector of α at s makes with v. Then $|\theta'(s)| = k(s)$.*

Proof. Since α is parametrized by arc length, $\|\alpha'(s)\| = 1$ and thus $\theta(s) = \cos^{-1}(\langle \alpha'(s), v \rangle)$. The use of $\cos^{-1}$ is justified if we assume θ to always give the smaller angle between the 2 vectors. Also let $n(s)$ denote the unit vector in the direction of $\alpha''(s)$. Then

$$\begin{aligned}
\theta'(s) &= \frac{-1}{\sqrt{1 - \langle \alpha'(s), v \rangle^2}} \cdot (\langle \alpha''(s), v \rangle + \langle \alpha'(s), 0 \rangle) \\
&= k(s) \frac{-\langle n(s), v \rangle}{\sqrt{1 - \cos^2(\theta(s))}} \\
&= k(s) \frac{-\langle n(s), v \rangle}{|\sin(\theta(s))|}.
\end{aligned}$$

Since α is parametrized by arc length, $\langle \alpha'(s), \alpha'(s) \rangle = 1$. Differentiating both sides shows that $\langle \alpha'(s), \alpha''(s) \rangle = 0$, so the normal and tangent vectors are orthogonal. So the lines spanned by v, $n(s)$, and $\alpha'(s)$ form a triangle, from which we conclude that the smaller angle between $n(s)$ and v equals $\pi/2 - \theta(s)$ (see Figure 3). Then:

$$\begin{aligned}
|\theta'(s)| &= k(s) \frac{|\cos(\pi/2 - \theta(s))|}{|\sin(\theta(s))|} \\
&= k(s) \frac{|\sin(\theta(s))|}{|\sin(\theta(s))|} \\
&= k(s).
\end{aligned}$$

□

Example 2.8. Recall that in Example 2.5 we showed that the equation for the line $\alpha(t) = x_0 + xt$ parametrized by arc length is $\beta(s) = x_0 + xt_0 + xs/\|x\|$. $\beta''(s) = 0$, so a line has zero curvature. This makes sense since a line does not "turn" at any point.

Example 2.9. A circle of radius r can be parametrized by arc length as $\alpha(t) = (r\cos(t/r), r\sin(t/r))$, since $\|\alpha'(t)\| = \|(-\sin(t/r), \cos(t/r)\| = 1$. We can then compute its curvature to be $k(s) = \|\alpha''(t)\| = \|(-1/r\cos(t/r), -1/r\sin(t/r))\| = 1/r$. This makes sense since the larger a circle is, the less sharp its perceived bend.

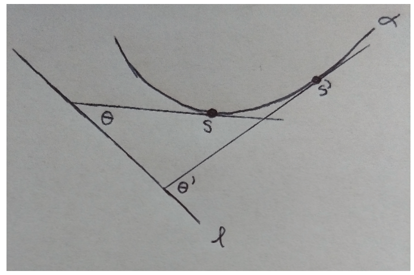


Figure 2: Angles made by tangent lines of a curve with a reference line ℓ

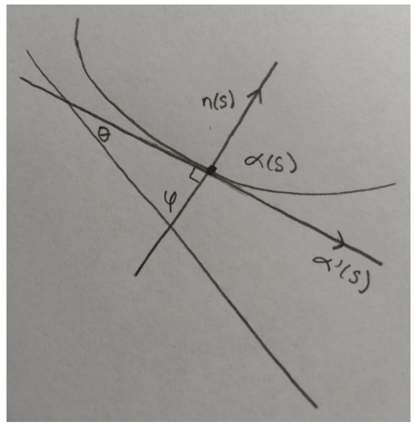


Figure 3: Depiction of the angle φ between normal vector and a reference line

2.3 The Osculating Circle

Definition 2.10. Let $\alpha : I \to \mathbb{R}^n$ be a curve and $p := \alpha(t) \in \mathbb{R}^n$ be a point on this curve with nonzero curvature. Let $\alpha(t_1) = p_1$ and $\alpha(t_2) = p_2$ be 2 other points on the curve with $t_1 < t < t_2$ sufficiently close to p that the 3 points are not colinear. A circle is uniquely determined by these 3 points. The *osculating circle* at a point p on the curve is the limiting circle obtained as p_1 and p_2 approach p. Figure 4 illustrates this limit.

The osculating circle at a point of nonzero curvature can be thought of as the closest-fitting tangent circle to that point on the curve. As expected, the curvature of the circle at that point equals the curvature k, so its radius $r = 1/k$ is called the *radius of curvature*. The reason that the definition requires α to have nonzero curvature at p is that otherwise the osculating circle could have an infinite radius.

The notion of an osculating circle provides a useful second-derivative analog of the tangent line at a point in the curve. A tangent line to a curve represents the path a car traveling along a curved road would take if it suddenly stopped following the road and started moving in a straight line at a point. Similarly, the osculating circle represents the circular path the car would take if it stopped following the road but "froze" the steering wheel's position at a point.

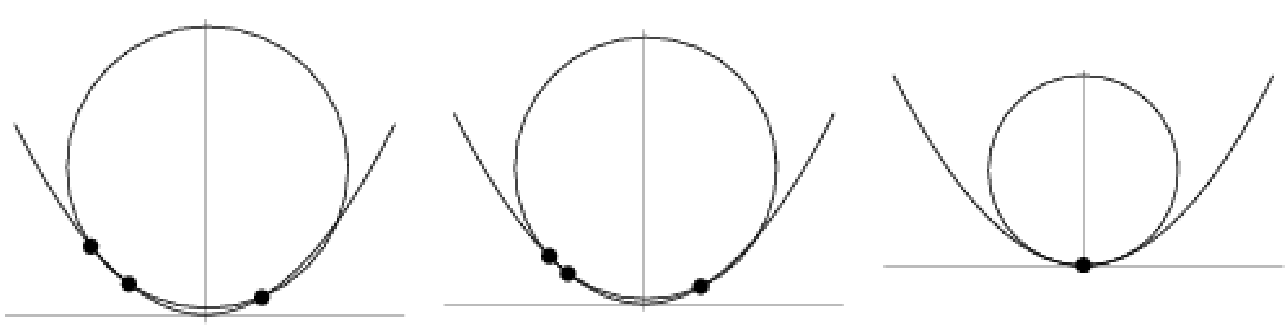


Figure 4: Osculating "limiting" circle at a point

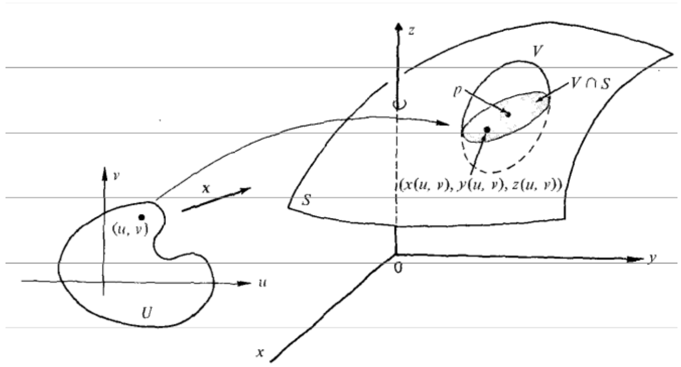


Figure 5: Illustration of the definition of a regular surface

3 Surfaces

3.1 Basic Definitions

The definition of a *regular surface* in $\mathbb{R}^3$ (to which we restrict our discussion) contains many parts that obfuscate its physical meaning. A regular surface is just a smooth 2-dimensional surface in $\mathbb{R}^3$ that looks flat when zoomed in, that is, it does not contain discontinuities or sharp points. Regular surfaces are a special case of *manifolds* in Euclidean space.
The following are examples of regular surfaces:

- Plane
- Sphere
- Torus
- Graph of any differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ (e.g. the paraboloid $f(x,y) = x^2 + y^2$)

The following are not examples of regular surfaces:

- Cone (because of the sharp tip)
- Line (since it is 1-dimensional and not locally planar)

Definition 3.1. A subset $S \subset \mathbb{R}^n$ is a *regular surface* if for every $p \in S$, there exists an open set $U \subset \mathbb{R}^2$ and a neighborhood of p, $V \subset \mathbb{R}^n$, such that there exists a map $x : U \to V \cap S$ such that:

- x is smooth (infinitely differentiable)
- x is a homeomorphism (a continuous bijection with a continuous inverse)
- the derivative of x at every point is injective

The above definition is illustrated in Figure 5. The map x at each point on the surface is considered a *parametrization* of the surface around the point. The first 2 conditions of x in the definition are used to enforce the "smoothness" property of regular surfaces. The last condition requires a bit more discussion. Note that $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, so its differential at a point q in U is just a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, which can be expressed as a 3×2 matrix. By the rank-nullity theorem, $\dim \text{Ker } dx_q + \dim \text{Im } dx_q = \dim \mathbb{R}^2 = 2$, so if dx_q is injective, then its image is 2-dimensional. Thus, this last condition is what makes regular surfaces locally planar.

Example 3.2. We show that the unit sphere is a regular surface. Recall that a unit sphere centered at the origin is the set of points $(x, y, z) \in \mathbb{R}^3$ that are unit distance away from the origin, that is, $x^2 + y^2 + z^2 = 1$. This means that the square of each individual x, y, or z is ≤ 1, so we can restrict our domain to $[-1, 1]^3$. Solving for z gives us $z = \pm\sqrt{1 - x^2 - y^2}$, and restricting z to be non-negative yields $z = \sqrt{1 - x^2 - y^2}$. The definition of a regular surface requires a map from an *open* subset of $\mathbb{R}^2$, so we define $f : (-1, 1)^2 \to \mathbb{R}^3$ as $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. This function is clearly one-to-one and thus in bijection with its image, which is the top half of the the unit sphere, excluding the circle in the xy plane.

The first criterion, that f is differentiable, is easily verified. The inverse of f is simply $f^{-1}(x, y, z) = (x, y)$, which is continuous. Lastly, we check that the differential of this map is one-to-one, meaning its matrix has linearly independent columns. The differential is as follows:

$$df = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{bmatrix}.$$

This is clearly true, so f is a local parametrization for any point in the Northern hemisphere of the unit sphere. The exact same construction of f can be modified to get parametrizations of 5 other hemispheres that together cover the unit sphere, as shown in Figure 6. Thus every point on the unit sphere has a local parametrization, meaning it is a regular surface.

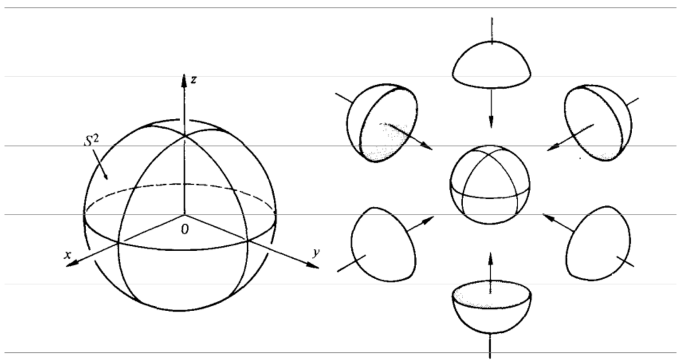


Figure 6: Partition of unit sphere into 6 hemispheres

3.2 The Tangent Plane and Gauss Map

We can easily generalize the notion of *the* tangent line to a curve to that of *a* tangent line at a point on a regular surface. A vector $v \in \mathbb{R}^3$ is tangent to a point p on a regular surface S if for some regular curve $\alpha : I \subset \mathbb{R} \to S$ (where I is an open interval containing 0), $\alpha(0) = p$ and $\alpha'(0) = v$.

Theorem 3.3. *Let S be a regular surface and $p \in S$. Let x be a parametrization of S at p, and $q = x^{-1}(p)$. Then the image of dx_q is the set of vectors tangent to p. The plane spanned by these vectors containing p is called the tangent plane to S at p.*

Proof. Let v be a tangent vector at p. Then for some curve $\alpha : I \to S$, $\alpha(0) = p$ and $\alpha'(0) = v$. Define the curve $\beta : I \to U$ as $\beta = x^{-1} \circ \alpha$. Then $dx_q(\beta'(0)) = dx_{\beta(0)}(\beta'(0)) = (x \circ \beta)'(0) = (x \circ x^{-1} \circ \alpha)'(0) = v$. Thus v is in the image of dx_q.

Now suppose $v = dx_q(w)$, that is, v is in the image of dx_q. Then define the curve $\alpha(t) = q + wt$. For small enough t, $q + wt \in U$ since U is an open subset of $\mathbb{R}^2$ containing q, so we can define a curve $\beta = x \circ \alpha$ such that $\beta : (-\varepsilon, \varepsilon) \to S$ for sufficiently small ε. Then $\beta(0) = p$ and $\beta'(0) = (x \circ \alpha)'(0) - dx_{\alpha(0)}(\alpha'(0)) = dx_q(w) = v$. Thus v is a tangent vector to S at p. □

Figure 7 shows an example of a tangent plane. The tangent plane at a point can be thought of as the plane that just touches that point on the surface and no other (within a neighborhood).

Example 3.4. We now determine the tangent plane of the "North pole" of the unit sphere (point $(0, 0, 1)$). From a previous example we know that a parametrization of the top half of the sphere is given by $f : (-1, 1)^2 \to \mathbb{R}^3, f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. $f(0, 0) = (0, 0, 1)$, and the differential of f at that point is

$$df_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So $(1, 0, 0)$ and $(0, 1, 0)$ are basis vectors for the image of this matrix, so the tangent plane at $(0, 0, 1)$ is the plane passing through that point and parallel to the xy plane. This tangent plane is shown in Figure 8.

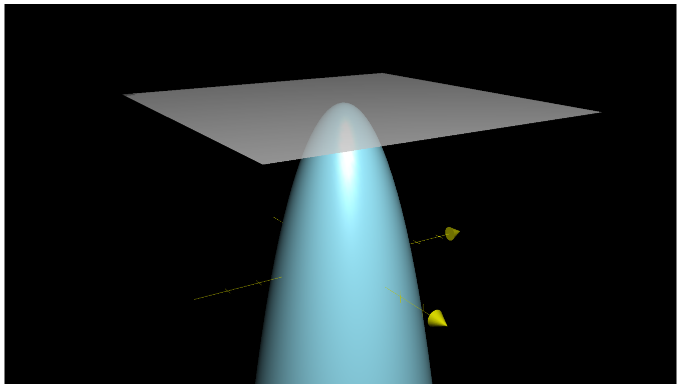


Figure 7: Example of tangent plane to a point on a surface

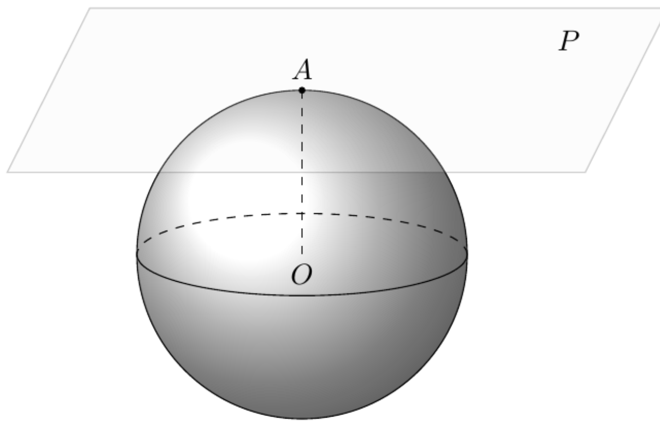


Figure 8: Tangent plane to North pole of unit sphere

The differential dx_q also provides a natural basis for the tangent plane of S at p, namely its 2 columns in matrix form. If the variables u and v are used to represent points in U, then $x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ where the x_i are smooth real-valued functions. Then the matrix of dx_q is

$$dx_q = \begin{bmatrix} \frac{\partial x_1}{\partial u}(q) & \frac{\partial x_1}{\partial v}(q) \\ \frac{\partial x_2}{\partial u}(q) & \frac{\partial x_2}{\partial v}(q) \\ \frac{\partial x_3}{\partial u}(q) & \frac{\partial x_3}{\partial v}(q) \end{bmatrix}.$$

We denote the 2 columns (and basis of the tangent plane) as x_u and x_v for short. Since the cross product of 2 vectors gives a vector orthogonal to both inputs, we can use the cross product to pick the unique unit vector (up to sign) that's normal to a regular surface at every point. If such a function is well-defined and continuous, the surface is called *orientable*. An orientable surface can be thought of as a surface with 2 well-defined sides (or an inside and outside).

A famous example of a non-orientable surface is a Möbius band. A Möbius band is not orientable since if we pick a unit normal vector at a point and trace a path along the band without "flipping" the direction of the normal vector, we arrive at the same point but with the normal vector pointing in the opposite direction. This is demonstrated in Figure 9.

Definition 3.5. Let S be an orientable regular surface and S^2 the unit 2-sphere. The unit normal vector at every point p can be identified as a point on S^2. Thus, we define the *Gauss map* $N : S \to S^2$ as:

$$N(p) = \frac{x_u \times x_v}{\|x_u \times x_v\|}$$

where x_u, x_v are basis vectors for the tangent plane at p, as determined by some parametrization x.

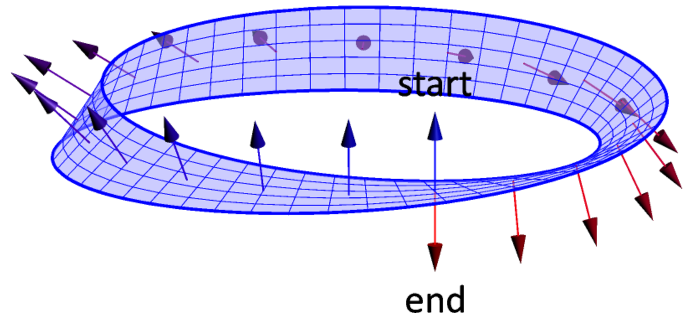


Figure 9: Non-orientability of the Möbius band

3.3 Normal Curvature

We now tackle the problem of defining curvature on regular surfaces. We do this by expressing it in terms of the curvatures of regular curves on the surface. We start with the notion of normal curvature, which is defined with respect to a given regular curve on a surface.

Definition 3.6. Let C be a regular curve on regular surface C passing through point $p \in S$, k be the curvature of C at p, n be the unit normal vector to C at p, N be the unit normal vector to the surface at p, and $\cos\theta = \langle n, N\rangle$. Then the *normal curvature* of C at point p is defined to be the *signed* quantity $k_n = k\cos\theta$.

At first glance, this definition doesn't seem terribly useful, since we'd like a definition of curvature that only depends on the properties of the surface, independent of the curves we can draw on it. It turns out that the definition of normal curvature is independent of the specific choice of curve C and only depends on the value of its tangent at point p. To do this, we first express the normal curvature in terms of the differential of the Gauss map. $N \circ C$ denotes the restriction of the Gauss map to the curve C. Since the normal vector $N(p)$ is orthogonal to every tangent vector at p, $\langle N(p), C'(0)\rangle = \langle N(C(0)), C'(0)\rangle = 0$. Differentiating both sides yields $\langle (N \circ C)'(0), C'(0)\rangle = -\langle N(p), C''(0)\rangle$. Thus,

$$\begin{aligned} k_n &= k(0)\cos(\theta) \\ &= k(0)\langle n(0), N(C(0))\rangle \\ &= \langle C''(0), N(C(0))\rangle \\ &= -\langle C'(0), (N \circ C)'(0)\rangle \\ &= -\langle C'(0), dN_p(C'(0))\rangle. \end{aligned}$$

The last line thus shows that the normal curvature only depends on the tangent $C'(0)$. This development is very reassuring, since it gives us a notion of the curvature of a surface *in a specific direction* in the tangent plane at p, namely $C'(0)$.

A natural next step would be to determine the directions of minimum and maximum normal curvature, and if a minimum and maximum even exist. It turns out that they do, and that they are the negative eigenvalues of the differential of the Gauss map at point p. We leave this unproven, but the interested reader can consult chapter 3.2 and the Appendix of Chapter 3 in do Carmo's textbook for more details.

Theorem 3.7. *Let S be a regular surface and $p \in S$. There exists an orthonoromal basis (e_1, e_2) of the tangent plane at p such that $dN_p(e_1) = -k_1 e_1$ and $dN_p(e_2) = -k_2 e_2$. Suppose w.l.o.g that $k_1 \geq k_2$. Then k_1 is the maximum normal curvature and k_2 is the minimal normal curvature at p.*

Definition 3.8. k_1 and k_2 in the previous theorem are called the *principal curvatures* of S at p. Their associated directions e_i are called the *principal directions*.

Example 3.9. Let S denote the half-pipe shown in Figure 10, whose cross-section is a semi-circle with radius r, and let p denote the point shown in the image. An informal examination of the surface tells us that the normal curvature doesn't change sign in any direction from p. The maximum magnitude of curvature is in the direction perpendicular to the axis of symmetry (e_1 in the figure), as one curve with that tangent line is just the semi-circle itself, which has curvature $1/r$.

Any curve passing through p whose tangent line at p is parallel to the pipe's axis of symmetry has a curvature of 0 at that point, since one such curve is a straight line, and normal curvature is independent of the specific choice of

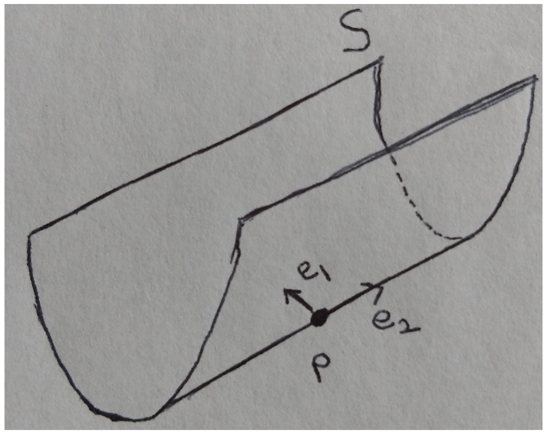


Figure 10: Half-pipe with semi-circular cross-section

curve. Since the normal curvature does not change sign in any direction from p, either the minimum or maximum normal curvature at p is 0 (depending on the chosen direction of the normal at that point). Thus the half-pipe has Gaussian curvature of 0 and is non-planar, meaning it is parabolic.

It turns out that the normal curvature in any direction can be calculated from the principal curvature and direction, and this fact is summed up in the following result, known as Euler's Theorem.

Theorem 3.10. *Let S be a regular surface and $p \in S$. Let k_1, k_2 be the minimum and maximum normal curvatures at p and e_i their associated principal directions. Then let v be some unit vector in $\mathbb{R}^2$. Then for some $\theta \in [0, 2\pi)$, $v = e_1 \cos\theta + e_2 \sin\theta$ and the normal curvature in the direction v is given by*

$$k_n = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Proof. Since the principal directions (e_1, e_2) form an orthonormal basis of $\mathbb{R}^2$, any unit vector v can be expressed in the form $e_1 \cos\theta + e_2 \sin\theta$ for some θ. This is the same way that any unit vector–identified as a point on the unit circle–has an angle $0 \leq \theta < 2\pi$ associated with it. Furthermore, $dN_p(e_1) = -k_1 e_1$ and $dN_p(e_2) = -k_2 e_2$.

From our previous expression of normal curvature k_n in terms of dN_p:

$$\begin{aligned}
k_n &= -\langle dN_p(v), v\rangle \\
&= -\langle dN_p(e_1\cos\theta + e_2\sin\theta), e_1\cos\theta + e_2\sin\theta\rangle \\
&= -\langle -k_1 e_1 \cos\theta - k_2 e_2 \sin\theta, e_1\cos\theta + e_2\sin\theta\rangle \\
&= k_1\cos^2\theta\langle e_1, e_1\rangle + k_1\cos\theta\sin\theta\langle e_1, e_2\rangle + k_2\cos\theta\sin\theta\langle e_1, e_2\rangle + k_2\sin^2\theta\langle e_2, e_2\rangle \\
&= k_1\cos^2\theta + k_2\sin^2\theta.
\end{aligned}$$

□

Thus the principal normal curvatures and principal directions succinctly provide us all the information about the normal curvature at a point. Our next step is to observe 2 important expressions of the principal curvatures–mean curvature and Gaussian curvature–and study what they tell us about a surface at a point.

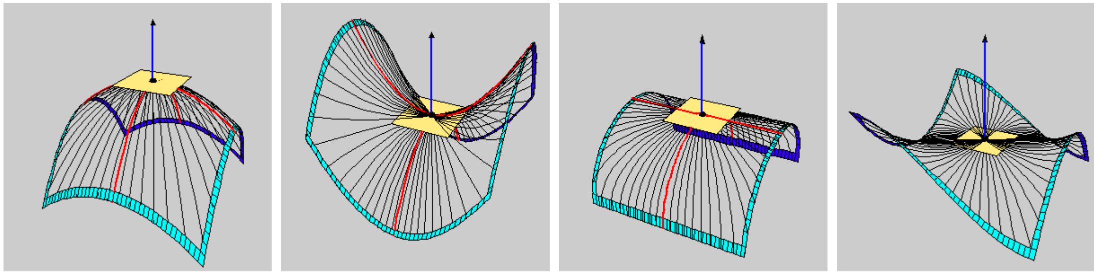
3.4 Mean and Gaussian Curvature

Recall that the trace and determinant of a linear operator are the sum and product of its eigenvalues respectively. These 2 properties of the linear map dN_p, the differential of the Gauss map at a point on a regular surface, are used in 2 important definitions of curvature.

Definition 3.11. Let S be a regular surface, and p be a point on this surface. Let k_1 and k_2 denote the principal curvatures (and negative eigenvalues of dN_p). The *Gaussian curvature* K of S at p is the determinant of dN_p. The *mean curvature* H of S at P is negative one-half of the trace of dN_p. These can be summarized as follows:

$$K = k_1 k_2,$$

$$H = \frac{k_1 + k_2}{2}.$$



(a) Elliptic (b) Hyperbolic (c) Parabolic (d) Planar

Figure 11: Classification of points by Gaussian curvature

Mean curvature is aptly named since it represents the "average normal curvature" over all directions. Recall from the proof of Euler's theorem that every unit vector $v \in \mathbb{R}^2$ can be written as $v = e_1 \cos\theta + e_2 \sin\theta$, where the e_i are the principal directions at a point p on regular surface S. We can thus express the normal curvature at p as a function of the angle which v makes with the basis e_i, using Euler's Theorem: $k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta$. We then compute the average value of this function for $\theta \in [0, 2\pi)$:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} k_n(\theta)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} k_1\cos^2\theta + k_2\sin^2\theta\, d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi}(k_1+k_2) + (k_1-k_2)\cos(2\theta)\, d\theta \\ &= \frac{k_1+k_2}{2} \\ &= H.\end{aligned}$$

The interpretation of Gaussian curvature is less obvious but can be understood by examining surfaces whose Gaussian curvatures have different signs.

Definition 3.12. Let S be a regular surface and $p \in S$. Let K denote the Gaussian curvature of S at p. This point is called:

- *Elliptic* if $K > 0$
- *Hyperbolic* if $K < 0$
- *Parabolic* if $K = 0, dN_p \neq 0$
- *Planar* if $K = 0, dN_p = 0$

We can informally think of an elliptic point as curving "the same way" in all directions (the way a sphere does) so that both principal curvatures are of the same sign. A hyperbolic point is one in which the principal curvatures are of opposite signs, so that it resembles a saddle point. A parabolic point has a positive principle curvature in some direction (since the surface is non-planar) but achieves a minimum curvature of 0 in some direction. In a neighborhood of a parabolic point, a surface looks like a curved piece of paper. In a neighborhood of a planar point, a surface just looks flat. Examples of all 4 types of points can be seen in Figure 11.

3.5 Theorema Egregium

In this section we see why Gaussian curvature is such a useful measure of the curvature of surfaces. Gaussian curvature's most useful property is that it is invariant under local isometries, which we formally define below.

Definition 3.13. A diffeomorphism φ (smooth homeomorphism) between regular surfaces S and $\bar{S}$ is an *isometry* if for all $p \in S$ and w_1, w_2 in the tangent space at p:

$$\langle w_1, w_2\rangle = \langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle.$$

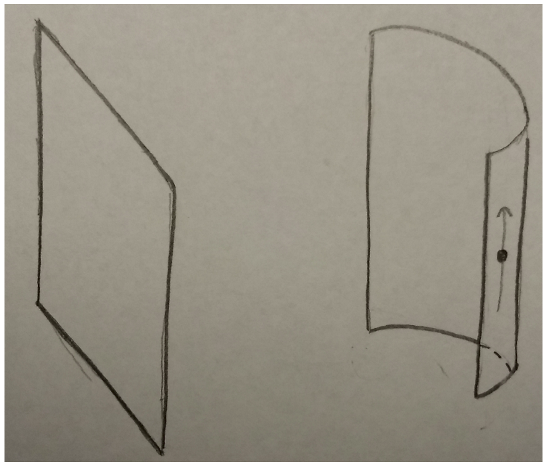


Figure 12: “Bending” isometry, with principal direction of zero curvature shown in the bent case

(a) Floppy pizza slice (b) Bent pizza slice, suitable for consumption

Figure 13: The correct way to hold a pizza slice

An isometry can be thought of as a map between surfaces that preserves the distances between points when traveling on the surface. A diffeomorphism that’s an isometry thus smoothly deforms a surfaces without stretching, shrinking, or poking/filling holes in it. A *local* isometry is just a map such that its restriction to a neighborhood around every point on the original surface is an isometry.

We now present Gauss’ Theorema Egregium (“Remarkable theorem”), an important result in differential geometry, followed by a couple informal examples:

Theorem 3.14. *The Gaussian curvature K of a regular surface is invariant under local isometries. Suppose we have regular surfaces S and $\bar{S}$ and a diffeomorphism $\varphi : S \to \bar{S}$ such that for every $p \in S$, there exists a neighborhood $U \subset S$ around p such that φ restricted to U is an isometry. Then the Gaussian curvature of S at p equals the Gaussian curvature of $\bar{S}$ at $\varphi(p)$.*

Example 3.15. Let S denote a plane (think of this as a sheet of paper). Its Gaussian curvature is clearly 0 everywhere. Bending this plane is a diffeomorphism that’s also an isometry since the distance between any 2 points on the sheet of paper (when traveling on the sheet of paper) is preserved. At every point on the bent sheet of paper, one of the principal curvatures is positive and the other is negative, as shown in Figure 12. Thus the Gaussian curvature of the bent paper is 0 at every point.

Example 3.16. Gaussian curvature can also be used to effectively hold a pizza slice when eating it. One common problem when eating a slice of pizza (especially a large one from, say, Costco) is that it sags, as shown in Figure 13. Any bending or folding done to the pizza slice is local isometry and thus preserves the Gaussian curvature at every point. Thus, folding the pizza slice along its axis of symmetry (as shown in Figure 13) increases its normal curvature in the direction perpendicular to the axis of symmetry and thus decreases the curvature in the direction parallel to the axis of symmetry, making the slice straighter in that direction and easier to eat.

4 Next Steps

In these notes, we defined the basic building blocks of differential geometry in Euclidean space, namely regular curves and surfaces, and then explored various definitions of curvature on these objects, beginning with the definition of curvature of curves and ending with the notion of Gaussian curvature, which has the "remarkable" property of being invariant under local isometries.

One limitation of this exposition is that we only worked in 3-dimensional Euclidean space. There are numerous fields in which geometry is done on much higher-dimensional spaces, such as in physics (in which Einstein's theory describes the universe as a 4-dimensional spacetime manifold) and machine learning (in which data points often contain 100s or 1000s of features). In these higher-dimensional settings, definitions must be generalized accordingly, such as how the notion of a normal vector generalizes to a vector subspace.

Geometry can be generalized even further, not only to higher-dimensional Euclidean spaces, but to abstract topological spaces. The definition of a regular surface generalizes to that of a smooth manifold, which is a topological space that's locally diffeomorphic to Euclidean space. The manifolds which provide us with an inner product, that is, a ruler and protractor with which geometry can be done, are called Riemannian manifolds, and the vast field of Riemannian geometry deals with these interesting spaces.

5 Figures Cited

The figures not listed below were hand-drawn by Ashwin Devaraj.

- Figure 4: https://mathworld.wolfram.com/OsculatingCircle.html
- Figure 5: *Differential Geometry of Curves and Surfaces* by do Carmo
- Figure 6: *Differential Geometry of Curves and Surfaces* by do Carmo
- Figure 7: https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/tangent-planes-and-local-linearization/a/tangent-planes
- Figure 8: https://tex.stackexchange.com/questions/373308/tikz-comparison-of-path-on-sphere-to-associated-path-in-the-tangent-plane
- Figure 9: https://sites.und.edu/timothy.prescott/apex/web/apex.Ch15.S5.html
- Figure 11: http://www.grad.hr/itproject_math/Links/sonja/gausseng/ehpp/ehpp.html
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